Lecture XVII: Weakly Interacting Electron Gas: Plasma Theory

▷ How are the properties of an electron gas influenced by weak Coulomb interaction?

▷ Qualitative considerations:

When is the intereaction weak? Defining $r_0 = \frac{1}{n^{1/3}}$ as the average electron separation, the typical p.e. $\frac{e^2}{r_0}$ and k.e. $\frac{\hbar^2}{mr_0}$ lead to the dimensionless ratio, $r_s = \frac{e^2}{r_0}\frac{mr_0^2}{\hbar^2} \equiv \frac{r_0}{a_0}$, where a_0 is electron Bohr radius, from which one can infer that Coulomb effects dominate at low density

At $r_s \sim 35$ there is (believed to be) a transition to an electron solid phase known as a Wigner crystal (cf. Mott-Hubbard insulator)

For most metals ($2 < r_s < 6$), k.e. and p.e. comparable; fortunately (thanks to adiabatic continuity) "weak coupling" theory valid even for intermediate r_s

▷ Motivates consideration of weak coupling theory $r_s \ll 1$: Σ-convention on spin

$$\hat{H} = \int d^d r\, c^\dagger_\sigma(\mathbf{r})\frac{\hat{\mathbf{p}}^2}{2m}c_\sigma(\mathbf{r}) + \frac{1}{2}\int d^d r \int d^d r' c^\dagger_\sigma(\mathbf{r})c^\dagger_{\sigma'}(\mathbf{r}')\frac{e^2}{|\mathbf{r}-\mathbf{r}'|}c_{\sigma'}(\mathbf{r}')c_\sigma(\mathbf{r})$$

Aim: to explore dielectric properties and ground state energy of electron gas through...

▷ Quantum partition function: using CSPI formulation

$$\mathcal{Z} \equiv \text{tr}\, e^{-\beta(\hat{H}-\mu\hat{N})} = \int_{\substack{\bar{\psi}_\sigma(0)=-\bar{\psi}_\sigma(\beta)\\ \psi_\sigma(0)=-\psi_\sigma(\beta)}} D(\bar{\psi}_\sigma,\psi_\sigma)e^{-S[\bar{\psi}_\sigma,\psi_\sigma]}$$

$$S[\bar{\psi}_\sigma,\psi_\sigma] = \int_0^\beta d\tau \left[\int d^d r\, \bar{\psi}_\sigma(\mathbf{r},\tau)\left(\partial_\tau + \frac{\hat{\mathbf{p}}^2}{2m} - \mu\right)\psi_\sigma(\mathbf{r},\tau)\right.$$
$$\left. + \frac{1}{2}\int d^d r \int d^d r' \bar{\psi}_\sigma(\mathbf{r},\tau)\bar{\psi}_{\sigma'}(\mathbf{r}',\tau)\frac{e^2}{|\mathbf{r}-\mathbf{r}'|}\psi_{\sigma'}(\mathbf{r}',\tau)\psi_\sigma(\mathbf{r},\tau)\right]$$

Expressed in Fourier basis: $\psi_\sigma(\mathbf{r},\tau) = \frac{1}{\sqrt{L^3\beta}}\sum_{\mathbf{k},\omega_n} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_n\tau)}\psi_{\mathbf{k},\omega_n,\sigma}$

$$S = \int_0^\beta d\tau \left[\sum_{\mathbf{k}} \bar{\psi}_{\mathbf{k}\sigma}(\tau)\left(\partial_\tau + \epsilon_{\mathbf{k}} - \mu\right)\psi_{\mathbf{k}\sigma}(\tau) + \frac{1}{2L^d}\sum_{\mathbf{q}\neq 0}\frac{4\pi e^2}{\mathbf{q}^2}\rho_{\mathbf{q}}(\tau)\rho_{-\mathbf{q}}(\tau)\right]$$

where $\epsilon_{\mathbf{k}} = \frac{\hbar^2\mathbf{k}^2}{2m}$ and $\rho_{\mathbf{q}}(\tau) = \int d^d r\, e^{-i\mathbf{q}\cdot\mathbf{r}}\rho(\mathbf{r},\tau) \equiv \sum_{\mathbf{k}}\bar{\psi}_{\mathbf{k}\sigma}(\tau)\psi_{\mathbf{k}+\mathbf{q},\sigma}(\tau)$
(N.B. neutralising background $\rightsquigarrow$ exclusion of $\mathbf{q}=0$ from sum)

With the action quartic in fermionic fields ψ, $\mathcal{Z}$ can not be evaluated exactly

For weak interaction, $r_s \ll 1$, we could expand in Coulomb interaction:
$\rightsquigarrow$ Feynman diagram expansion (cf. Gell-Mann—Brückner theory)

Alternative — use field integral to isolate leading diagrammatic series expansion
— known as the Random Phase Approximation (RPA)

▷ <u>General principle:</u>

When confronted with interacting field theory, seek decomposition of interaction through introduction of auxiliary field which captures low-energy content of theory

In some cases, these fields are identified with the elementary particles that mediate the interaction (see below); in others, these fields encode the low-energy collective modes of the system (e.g. superfluid, superconductor)

▷ Decoupling facilitated using the <u>Hubbard-Stratonovich transformation</u>:

$$e^{-\int_0^\beta d\tau \sum_{\mathbf{q}\neq 0} \frac{2\pi e^2}{L^d \mathbf{q}^2}\rho_{\mathbf{q}}(\tau)\rho_{-\mathbf{q}}(\tau)} = \int D\phi\, e^{-\int_0^\beta d\tau \sum_{\mathbf{q}\neq 0}\left[\frac{\mathbf{q}^2}{8\pi}\phi_{\mathbf{q}}(\tau)\phi_{-\mathbf{q}}(\tau)+\frac{ie}{2L^{d/2}}(\phi_{\mathbf{q}}(\tau)\rho_{-\mathbf{q}}(\tau)+\rho_{\mathbf{q}}(\tau)\phi_{-\mathbf{q}}(\tau))\right]}$$

▷ Physically, ϕ represents (scalar) photon field which mediates Coulomb interaction
N.B. ϕ real and periodic $\phi(\tau+\beta)=\phi(\tau)$

$$\mathcal{Z} = \int D(\bar\psi_\sigma,\psi_\sigma)\int D\phi\ \exp\left\{-\int_0^\beta d\tau\int d^d r\left[\frac{1}{8\pi}(\partial\phi)^2+\bar\psi_\sigma\left(\partial_\tau+\frac{\hat{\mathbf{p}}^2}{2m}-\mu+ie\phi\right)\psi_\sigma\right]\right\}$$

Gaussian in Grassmann fields, field integral may be performed:
using identity $\int D[\bar\psi,\psi]\exp[-\bar\psi M\psi] = \det M = \exp[\ln\det M]$

$$\boxed{\mathcal{Z} = \int D\phi\exp\left[-\int_0^\beta d\tau\int d^d r\frac{1}{8\pi}(\partial\phi)^2 + \overbrace{2}^{\text{spin}}\ln\det\left(\partial_\tau+\frac{\hat{\mathbf{p}}^2}{2m}-\mu+ie\phi\right)\right]}$$

Setting $e=0$, photon field decouples from determinant;
recovers partition function of non-interacting electron gas

▷ <u>Perturbation Theory in e:</u>

Define free particle Green function: $\hat G_0 = [\partial_\tau + \frac{\hat{\mathbf{p}}^2}{2m}-\mu]^{-1}$ and expand:
$\ln(1+x) = -\sum_{n=1}(-x)^n/n$

$$\begin{aligned}\ln\det\left(\partial_\tau+\frac{\hat{\mathbf{p}}^2}{2m}-\mu+ie\phi\right) &\equiv \mathrm{tr}\ln\left(\hat G_0^{-1}+ie\phi\right) = \mathrm{tr}\ln\hat G_0^{-1} + \mathrm{tr}\ln\left[1+ie\hat G_0\phi\right]\\ &= \mathrm{tr}\ln\hat G_0^{-1} - \mathrm{tr}\left[-ie\hat G_0\phi+\frac{1}{2}\left(ie\hat G_0\phi\right)^2+\cdots\right]\end{aligned}$$

- First order term: for convenience, set $k\equiv(\mathbf{k},\omega_n)$, etc.

$$2\mathrm{tr}[\hat G_0\phi] = 2\sum_k \overbrace{\langle k|\hat G_0|k\rangle}^{G_0(k)}\ \overbrace{\langle k|\phi|k\rangle}^{\frac{1}{\sqrt{L^3\beta}}\phi_{k=0}} = \frac{2}{\sqrt{L^3\beta}}\sum_k\frac{1}{-i\omega_n+\epsilon_{\mathbf{k}}-\mu}\phi_0 = 0$$

$\phi_0 = 0$ due to neutralising background

- Second order term:

$$2\times\frac{e^2}{2}\mathrm{tr}[\hat{G}_0\phi]^2 = e^2\sum_{k,q}\overbrace{\langle k|\hat{G}_0|k\rangle}^{G_0(k)}\overbrace{\langle k|\phi|k+q\rangle}^{\frac{1}{\sqrt{\beta L^3}}\phi_q}\overbrace{\langle k+q|\hat{G}_0|k+q\rangle}^{G_0(k+q)}\overbrace{\langle k+q|\phi|k\rangle}^{\frac{1}{\sqrt{\beta L^3}}\phi_{-q}} = \frac{e^2}{2}\sum_q \Pi(q)\phi_{-q}\phi_q$$

where "density-density" response function,

$$\boxed{\Pi(q) = \frac{2}{\beta L^3}\sum_k \frac{1}{-i\omega_n + \epsilon_{\mathbf{k}} - \mu}\,\frac{1}{-i\omega_n - i\omega_m + \epsilon_{\mathbf{k+q}} - \mu}}$$

Combined with bare term, to leading order in e^2 (Random Phase Approximation),

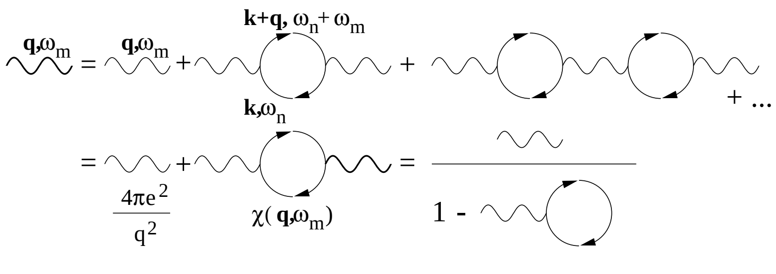
$$\mathcal{Z} = \mathcal{Z}_0\int D\phi\, e^{-S[\phi]}, \qquad S[\phi] = \frac{1}{2}\sum_q \overbrace{\left(\frac{\mathbf{q}^2}{4\pi} - e^2\Pi(q)\right)}^{D^{-1}(q)}|\phi_q|^2 + O(e^4)$$

$\mathcal{Z}_0$ denotes partition function of non-interacting gas

▷ Physically, $D^{-1}(q)$ denotes dynamically <u>screened Coulomb interaction</u>

$$\boxed{D^{-1}(q) = \epsilon(q)\frac{\mathbf{q}^2}{4\pi}, \qquad \epsilon(q) = 1 - \frac{4\pi e^2}{\mathbf{q}^2}\Pi(q)}$$

where $\epsilon(q)$ is the energy and momentum dependent effective <u>dielectric function</u>



Diagrammatic interpretation:

$$D(q) = \frac{4\pi}{\mathbf{q}^2}\frac{1}{1 - \frac{4\pi e^2}{\mathbf{q}^2}\Pi(q)} = \frac{4\pi}{\mathbf{q}^2}\sum_{n=0}^{\infty}\left(e^2\Pi(q)\frac{4\pi}{\mathbf{q}^2}\right)^n$$